GROMOV-WITTEN THEORY OF LOCALLY CONFORMALLY SYMPLECTIC MANIFOLDS AND THE FULLER INDEX

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Abstract. We set up a Gromov-Witten type theory of locally conformally symplectic manifolds. First we show using a theorem of P. Topping that the Kontsevich stable-map completed moduli spaces of pseudo-holomorphic curves in a locally conformally symplectic manifold (or l. c. s. m.) form non-compact but complete spaces (in the Gromov metric). To obtain (Gromov-Witten) invariants in this non compact setting, we propose that the energy function on the moduli space of holomorphic curves in an l. c. s. m. behaves in crucial examples like a bounded from below proper abstract moment map for an $T = S^1$ action, in the sense of Karshon, the Gromov-Witten invariant must then be considered in a suitable T equivariant sense. One basic example considered here, where this holds, is that of a locally conformally symplectic manifold $C \times S^1$ coming from a contact manifold (C, ξ). And using our theory we define a $\mathbb{Q}$ valued "quantum Euler characteristic of (C, ξ)" which "counts" Reeb orbits of (C, ξ) with respect to a contact form λ, in terms of certain Gromov-Witten invariants counting holomorphic tori in $C \times S^1$. Using this we give some new results on the existence of Reeb orbits, which are possibly not reproducible with contact homology techniques. This count is very closely related with the Fuller index of the Reeb vector field, which is a $\mathbb{Q}$-valued fixed point like index, for orbits of a general vector field. And Gromov-Witten theory suggests natural extensions of the Fuller index, as well as natural extensions of the Seifert conjecture for S^{2n+1}.

Contents

1. Introduction

A locally conformally symplectic manifold of dimension $2n$ is a smooth $2n$-fold M with a non-degenerate 2-form ω which is locally diffeomorphic to $e^f \omega_0$, for some (non-fixed) function f, with ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. These were originally considered by Lee in [11], arising naturally as part of an abstract study of "even" dimensional Riemannian geometry, and then further studied by a number of authors see for instance, [1] and [18]. This is a fascinating object, an l. c. s. m. admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and

Murphy show that if a closed almost complex $2n$-fold M has $H^1(M, \mathbb{R}) \neq 0$ then it admits a l. c. s. m. structure, [5], see also [2]. Such manifolds also appear naturally in some physical contexts.

However Gromov-Witten type theory of these manifolds, has not been considered. This is tricky, because in interesting examples there will usually not be any global Gromov compactness, in fact even the crucial energy quantization, which we prove here requires a substantial (although very intuitive) theorem of P. Topping.

A very interesting class of explicit examples of l. c. s. m.'s, is obtained by starting with a symplectic cobordism of a closed contact manifold C to itself, arranging for the contact forms at the two ends of the cobordism to be proportional (which can always be done) and then gluing together the boundary components. As a particular case of this we get Banyaga's basic example.

Example 1 (Banyaga). Let (C, ξ) be a contact manifold with a contact form λ and take $M = C \times S^1$ with 2-form $\omega = d^\alpha \lambda := d\lambda - \alpha \wedge \lambda$, for α the pull-back of the volume form on S^1 to $C \times S^1$ under the projection. Clearly for any pair of contact forms the resulting l. c. s. m. structures are T-equivariantly deformation invariant.

In the case of the example above, our Gromov-Witten invariants particularly something we call quantum Euler characteristic, turn out to be closely related to the Fuller index of the Reeb vector field, associated to a fixed free homotopy class of loops in C, and which in principle is defined under very general assumptions. In some very special cases, (finitely many Reeb orbits, and no multiply covered orbits in a fixed homotopy class), quantum Euler characteristic also coincides with the Euler characteristic of cylindrical contact homology, (when it is defined). In general it is tricky to relate quantum Euler characteristic to contact homology Euler characteristic. But hypothetically it may work like the following: if cylindrical contact homology was finite dimensional and functorially identified with rational homology of the quotient space Y/T, for a space Y equipped with a T action, (so that the quotient is of orbifold type) then quantum Euler characteristic that we define would be the orbifold Euler characteristic of the orbifold T quotient of Y.

We note that in the main example our invariants are counting holomorphic tori, and consequences for existence of Reeb orbits are obtained. This is exciting, as although there is much research on enumerative geometry, counting genus 1 curves, geometric applications of Gromov-Witten theory in symplectic geometry have to my knowledge been mostly concerned with genus 0 curve counts. In this note genus 1 curve counts will have a deep geometric significance.

Given a l. c. s. m. (M, ω), an almost complex structure J is called J-compatible if $\omega(\cdot, J\cdot)$ is a J-invariant metric on M. We shall denote this metric by g_J.

Theorem 1.1. *Let (M, ω) be a compact l.c.s. manifold. Let J be a compatible almost complex structure and let $u : \Sigma \to M$ be J-holomorphic map. Then there is an $\hbar > 0$ depending on ω, J s.t. if* energy$(u) < \hbar$ *then u is constant, where* energy *is L^2 energy.*

Remark 1.2. For the proof we may actually work with a priori more general notion of a l. c. s. m.. That is define a l. c. s. m. of fancy kind, to be a smooth $2n$-fold with a non-degenerate 2-form ω, s.t. for any $p \in M$ there is a chart (U, ϕ), $U \ni p$ s.t. for any ω-compatible almost complex structure J on U, $\phi_* J$ is compatible with the standard symplectic form on $\mathbb{R}^{2n}$. It follows that given such a structure the tangent transition maps will induce linear automorphisms of $\mathbb{R}^{2n}$ which take any ω compatible complex structure on the vector space $\mathbb{R}^{2n}$ (an endomorphism J of $\mathbb{R}^{2n}$ with $J^2 = -id$) to an ω compatible complex structure, for ω here the standard symplectic form on the vector space $\mathbb{R}^{2n}$. We claim (although this is not obvious, but is fairly elementary) that this group is the group of conformal linear symplectic automorphisms. It follows immediately from this that the notion of l. c. s. m. of fancy kind is equivalent to the classical notion.

Given the theorem above if we had universal L^2 energy bounds for J-holomorphic curves in a fixed class A we would immediately obtain Gromov compactness theorem for class A J-holomorphic curves in an l. c. s. m. (M, ω). However because ω is not closed, there are no a priori energy bounds as for symplectic manifolds, and in a general l. c. s. m. this energy could be unbounded. We shall

see this happen in our examples further on. Nevertheless given Theorem 1.1 the classical "Gromov compactness" argument immediately gives the following:

Theorem 1.3. *The moduli space $\mathcal{M}_g(J, A)$ of equivalence classes of J-holomorphic curves in class A, with domain a genus g, closed Riemann surface has a completion*

$$\overline{\mathcal{M}}_g(J, A),$$

by Kontsevich stable maps. The completion here is in the sense of the natural metrizable Gromov topology see for instance [15], for genus 0 case. Moreover given $E > 0$, the subspace $\overline{\mathcal{M}}_g(J, A)_E \subset \overline{\mathcal{M}}_g(J, A)$ consisting of elements with L^2 energy $\leq E$ is compact.

Thus the energy function energy gives a proper bounded from below function on the moduli space. The equivalence classes are as in the definition of $\overline{\mathcal{M}}_1(J, A)$, in Section 1.1. We shall show further on that in the setting of Example 1 energy is bounded on the connected components of the moduli space. Strictly speaking we show that it is bounded on components of $\overline{\mathcal{M}}_1(J, A)$, but then it is easy to see that every holomorphic curve in $C \times S^1$ is in fact a covering map of genus 1 curve, so that the case of a general curves is implied. This allows us to obtain interesting invariants even with energy being globally unbounded. In fact suitably interpreted energy in this example is an abstract, proper, bounded from below, moment map function in the sense of Karshon [9], for a specific circle action. This suggests a path to generalizing some of our constructions to more general examples of l. c. s. m.'s, endowed with a circle action, which will be explored elsewhere.

1.1. **Gromov-Witten theory of the** l. c. s. m. $C \times S^1$**.** Let (C, λ) be a closed contact manifold with contact form λ. Then $T = S^1$ acts on $C \times S^1$ by rotation in the S^1 coordinate. Let J be an almost complex structure on the contact distribution, compatible with $d\lambda$. There is an induced almost complex structure J^λ on $C \times S^1$, which is T-invariant, coincides with J on the contact distribution

$$\xi \subset TC \oplus \{\theta\} \subset T(C \times S^1),$$

for each θ and which maps the Reeb vector field

$$R^\lambda \in TC \oplus 0 \subset T(C \times S^1)$$

to

$$\frac{d}{d\theta} \in \{0\} \oplus TS^1 \subset T(C \times S^1),$$

for $\theta \in [0, 2\pi]$ the global angular coordinate on S^1. This almost complex structure is compatible with $d^\alpha \lambda$.

We shall be looking below at the moduli space of holomorphic tori in $C \times S^1$. Our notation for this is $\overline{\mathcal{M}}_1(J^\lambda, A)$, where A is a class of the maps. The elements are equivalence classes of pairs (u, Σ): u a J^λ-holomorphic map of a stable genus 1 curve Σ into $C \times S^1$. So Σ is a nodal curve with principal component a genus 1 curve, and other components spherical. We assume that the principal component is smooth, so determines an element of M_1 the moduli space of genus 1 Riemann surfaces, which is understood as an orbifold. The equivalence relation is $(u, \Sigma) \sim (u', \Sigma')$ if there is a biholomorphism $\phi : \Sigma \to \Sigma'$ s.t. $u' \circ \phi = u$. When Σ is smooth, we may write $[u, j]$ for an equivalence class where j is understood as a complex structure on the sole principal component of the domain, and u the map. Or we may just write $[u]$ keeping track of j implicitly. We do not need to compactify M_1, because nodal degenerations of the complex structure will be impossible in our specific geometric situation.

1.1.1. *Reeb tori.* For the almost complex structure J^λ as above we have one natural class of holomorphic tori in $C \times S^1$ that we call *Reeb tori*. Given a closed orbit o of R^λ, a Reeb torus u_o for o, is the map

$$u_o(\theta_1, \theta_2) = (o(\theta_1), \theta_2),$$

$\theta_1, \theta_2 \in S^1$ A Reeb torus is J^λ-holomorphic for a uniquely determined holomorphic structure j on T^2. If

$$D_t o(t) = c \cdot R^\lambda(o(t)),$$

then
$$j(\frac{\partial}{\partial\theta_1}) = c\frac{\partial}{\partial\theta_2}.$$

Proposition 1.4. *Let (C, λ) be as above. Let A be the homology class of a map $T^2 \to C \times S^1$ with "degree" 1 projection to S^1, and J^λ be as above. Then the entire moduli space is is fixed by the T action, and consists of Reeb tori. For a pair $J^{\lambda_0}, J^{\lambda_1}$ of almost complex structures as above and $\{J^{\lambda_t}\}$ a smooth interpolating family, $\overline{\mathcal{M}}_1(\{J^{\lambda_t}\}, A)$, has compact connected components.*

The basis for the second part of the proposition is the following crucial lemma.

Lemma 1.5. *Let $\{R^{\lambda_t}\}$, $0 \leq t \leq 1$, be a family of Reeb vector fields corresponding to a smooth family $\{\lambda_t\}$ of contact forms on a closed contact manifold C. Then all the connected components of the solution space*
$$S = \{(o, t) \,|\, o \text{ is a periodic orbit of } R^{\lambda_t}\}$$
are compact.

We shall say more on this property of Reeb vector fields in Section 1.3.

Note that the formal dimension of $\overline{\mathcal{M}}_1(J^\lambda, A)$ is 0, for A as in the proposition above. It is given by the Fredholm index of the operator (3.7) which is 2, minus the dimension of the automorphism group (for smooth curves) which is 2.

Proposition 1.6. *Let (C, ξ) be a general contact manifold. If λ is non-degenerate then all the elements of $\overline{\mathcal{M}}_1(J^\lambda, A)$, are regular curves. Moreover if λ is degenerate then for a period P Reeb orbit o the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus of o is identified with the 1-eigenspace of $\phi^\lambda_{P,*}$ - the time P linearized return map $\xi(o(0)) \to \xi(o(0))$ induced by the R^λ Reeb flow.*

Proposition 1.7. *Let (C, λ) be a contact $2n+1$-fold, and o a non-degenerate, period P, R^λ-Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}_1(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*
$$\operatorname{sign}\operatorname{Det}(\operatorname{Id}|_{\xi(o(0))} - \phi^\lambda_{P,*}|_{\xi(o(0))}).$$

1.2. **Quantum Euler number of (C, ξ).** Although we have a very strong transversality criterion in the form of Proposition 1.6, so that we do not need virtual moduli cycle techniques to regularize the moduli space $\overline{\mathcal{M}}_1(J^\lambda, A)$, it seems that we do need the virtual moduli cycle to regularize cobordisms of these moduli spaces. Indeed if one takes the "cobordism" of the type
$$\overline{\mathcal{M}}_1(\{J^{\lambda_t}\}, A) := \bigsqcup_t \overline{\mathcal{M}}_1(J^{\lambda_t}, A),$$
for a homotopy $\{\lambda_t\}$ then there will generically be bifurcations in the space of periodic orbits of the vector fields R^{λ_t}, since these correspond to our curves generically $\overline{\mathcal{M}}_1(\{J^{\lambda_t}\}, A)$ will not be regular. Moreover one should keep in mind that there are multiply covered curves, which must be counted appropriately to get invariants, see Section 2. Formally our moduli spaces $\overline{\mathcal{M}}_1(J^\lambda, A)$ are 0-dimensional non-effective orbifolds, the "count" of elements is the orbifold Euler number of these orbifolds.

Definition 1.8. *Let (C, ξ) be a contact, closed $2n+1$-fold, which admits a contact form λ whose Reeb orbits in class $[o]$ have bounded period. Let A be the class of a Reeb torus corresponding to $[o]$ as above. We define $\mathcal{E}(C \times S^1, A) = \#\overline{\mathcal{M}}_1(J^\lambda, A)$ where $\#\overline{\mathcal{M}}_1(J^\lambda, A)$ is the Gromov-Witten invariant:*
$$\int_{[\overline{\mathcal{M}}_1(J^\lambda, A)]^{vir}} 1,$$
see Section 2.1. (We could just ask that λ is non-degenerate above but then lose enormous flexibility that is provided by working with abstract perturbations as opposed to λ perturbations, in particular the Morse-Bott case becomes trivial, from the above view point.)

Otherwise we set $\mathcal{E}(C \times S^1, A)$ to be $\pm\infty$, in the case when we have a non-degenerate λ, and there is an $E > 0$, so that all the elements of $\overline{\mathcal{M}}_1(J^\lambda, A)$, with energy $> E$ *are positively, respectively*

negatively signed. Or if λ is degenerate, then the connected components $\mathcal{C}_i$ of $\overline{\mathcal{M}}_1(J^\lambda, A)$, are compact, by Proposition 1.4. We then ask that there are choices of abstract perturbations, so that there is an $N > 0$, s.t. for every $i > N$ all the elements of $\mathcal{C}_i^{vircycle}$, are positively, respectively negatively signed, see Section 2.2. (As in the bounded case working with abstract perturbations can be much simpler than directly perturbing λ.) We shall call the possibilities for λ: bounded*, respectively* infinite *(positive, negative) type.*

The above invariant is in a sense T-equivariant because it is a priori invariant only under very special T-equivariant deformations of the pair $(d^\alpha\lambda, J^\lambda)$ structure, namely the deformations corresponding to deformations of λ. This is the subject of the following lemma. We claim that they are in fact invariant under more general T-equivariant deformations, in a way that is connected to the theory of Karshon [9] but we postpone this story.

Lemma 1.9. *The invariant $\mathcal{E}(C \times S^1, A)$ is well defined, (in the cases it is defined). That is independent of the choice of λ.*

Theorem 1.10. *For a contact closed $2n+1$-fold C, if λ is non-degenerate and bounded type in class $[o]$ then:*

$$\mathcal{E}(C \times S^1, A) = \sum_o \frac{1}{\mathrm{mult}(o)}(-1)^{CZ(o)-n},$$

where the sum is over all (unparametrized) R^λ Reeb orbits, in class $[o]$, A determined by $[o]$ as usual.

1.3. **Connection to the Fuller index.** The Fuller index is an analogue for orbits of the fixed point index. But with a couple of new ingredients: we must account for the symmetry of the orbits, and since the period is freely varying there is an extra compactness issue to deal with. Let us very briefly recall this notion following Fuller's original paper [8] as the connection of this index with our quantum Euler characteristic is remarkable.

Let X be a smooth vector field on M, for simplicity without zeros, where M for simplicity is closed. The setting for the Fuller index is the phase space $M \times \mathbb{R}_+$. Denote by ϕ_t the time t flow map for X. We want to consider equivalence classes of points $(p, P) \in M \times \mathbb{R}_+$ which satisfy $\phi_P(p) = p$, with $(p, P) \sim (p', P)$ if $p' = \phi_t(p)$ for $0 \leq t \leq P$. We call the equivalence classes $o = [(p, P)]$: periodic orbits. We shall denote by $\underline{o}$ the (S^1-reparemtrization equivalence class) of a periodic orbit in M underlying o. The multiplicity $\mathrm{mult}(o)$ of a periodic orbit $o = [(p, P)]$ is the ratio P/l for $l > 0$ the least number with $\phi_l(p) = p$. We want a kind of fixed point index which counts such pairs (p, P) with certain weights, however in general to get invariance we must have period bounds. This is due to potential existence of so called blue sky catastrophe, where one has $\{X_\tau\}$ periodic orbits $\{o_\tau\}$, with this family continuous in the loop space LM, with period of $o(\tau)$ going to infinity as $\tau \mapsto a$, there are then no X_a periodic orbits o_a for all sufficiently large values of the period. So in effect this is a "bifurcation" where the orbit disappears into the sky.

Let $N \subset M \times \mathbb{R}_+$ be a compact isolating neighborhood. That is there is an open neighborhood of N not containing periodic orbits of X, which are not in N. Assume that free homotopy class c periodic orbits of X are isolated. Then to such an X, N Fuller associates an index:

$$i(X, N) = \sum_{o=[(p,P)],(p,P)\in N,[\underline{o}]\in c} \frac{1}{\mathrm{mult}(o)} i(o),$$

where $i(o)$, is the fixed point index of the return map with respect to a local surface of section transverse to $\underline{o}$, and $[\underline{o}]$ denotes the free homotopy class. In the case where X is the R^λ-Reeb vector field on C, and if λ is non-degenerate we have:

$$i(o) = \mathrm{sign}\,\mathrm{Det}(\mathrm{Id}\,|_{\xi(\underline{o}(0))} - \phi^\lambda_{P,*}|_{\xi(\underline{o}(0))}).$$

Fuller then shows that $i(X, N)$ is invariant in a deformation $\{X_t\}$ of X if N is isolating for X_t for all t. (Actually Fuller shows that we may also suitably vary N.) Note that if X is a λ-Reeb vector field on C, λ non-degenerate and $N = C \times [a, b]$ is isolating then $i(X, N)$ is just the orbifold Euler number of the 0-dimensional orbifold $\overline{\mathcal{M}}_1(J^\lambda, A)_{a,b}$ consisting of elements of $\overline{\mathcal{M}}_1(J^\lambda, A)$ corresponding to Reeb

orbits o in class c with $a \leq \operatorname{period} o \leq b$. However Lemma 1.5 shows that sky catastrophes cannot happen in the contact setting. We then conjecture that the Fuller index has natural extensions (in the cases where λ is bounded or infinite type) to the whole phase space $C \times \mathbb{R}_+$ coinciding with the quantum Euler numbers $\mathcal{E}(C \times S^1, A)$. This of course also gives an elementary argument for invariance of $\mathcal{E}(C \times S^1, A)$ since the Fuller index is invariant. This (apparently) very innocuous conjecture has somewhat surprising consequences, in that the results on existence of Reeb orbits in Section 1.4 can be given an elementary proof using the Fuller index and consequently extend to much more general vector fields and isotopies of vector fields. Not completely general since in the least we must require that in an isotopy $\{X_t\}$ of a vector field there are no sky catastrophes. As a simple example we conjecture the following analogue of the Seifert conjecture:

Conjecture 1. *Let X be a non-singular C^∞ vector field on S^{2n+1} homotopic to the standard Reeb vector field through non-singular C^∞ vector fields $\{X_t\}$, $0 \leq t \leq 1$ such that all the connected components of the solution space*

$$S = \{(o,t) \,|\, o \text{ is a periodic orbit (in the usual sense) of } X_t\}$$

are compact. Then X admits a periodic orbit.

We emphasize that this applies in particular to homotopies through Reeb vector fields, but the proof would involve no holomorphic curves! Although we state this as a conjecture a proof of this claim is in preparation by the author to appear in another paper.

1.4. **Applications.** Recall the standard definition:

Definition 1.11. *A contact form λ is called* Morse-Bott *if the λ action spectrum $\sigma(\lambda)$ is discreet and if for every $A \in \sigma(\lambda)$, the space $N_a := \{p \in C | \, \phi_a(p) = p\}$, ϕ_a the time a Reeb flow - is a closed smooth manifold such that rank $d\alpha|_{N_a}$ is locally constant and $T_pN_a = \ker(d\phi_a - I)_p$. In this case the (generalized) Conley-Zehnder index, is locally constant and we shall denote by $N_{a,l}$ the submanifolds of N_a consisting of Reeb orbits with Conley-Zehnder index l.*

For the Morse-Bott case as in the definition above, the quotient spaces of each $N_{a,l}$ by the S^1 reparametrization group action, are smooth (non-effective) orbifolds $R_{a,l}$, which we call *Reeb orbifolds*. Notice that when we take the component in $\overline{\mathcal{M}}_1(J^\lambda, A)$ corresponding to $R_{a,l}$, the orbifold structures are identified, for while the complex structure on the domain of a curve $[u]$, can have additional symmetry it will not give rise to a symmetry of the curve if $[u]$ is non-constant. With this in mind we shall denote this corresponding component by $\widetilde{R}_{a,l}$.

Here is one corollary of Theorem 1.10.

Corollary 1.12. *Let C be a contact closed $2n+1$-fold, admitting a non-degenerate contact form λ of bounded type in class $[o]$, then for any other contact form on C in the homotopy class (through contact forms) of λ there must be at least*

$$\operatorname{ceil} |\sum_o \frac{1}{\operatorname{mult}(o)} (-1)^{CZ(o)-n}|$$

Reeb orbits in class $[o]$, where ceil *is the function on $\mathbb{R}_+$,* $\operatorname{ceil}(x)$ *is the least integer greater than x.*

Presumably, in this generality this out of reach of techniques of contact homology, but the author is not expert. The above has a generalization to the Morse-Bott case as follows.

Theorem 1.13. *Suppose that λ is a Morse-Bott contact form of bounded type in class $[o]$, so that there are finitely many connected components of the Reeb orbifolds. Suppose that the associated Cauchy-Riemmann operators are complex linear for each $[u] \in \widetilde{R}_{a,l}$. Then*

$$\mathcal{E}(C, A_o) = \sum_{a,l} (-1)\chi(\widetilde{R}_{a,l}) = \sum_{a,l} (-1)\chi(R_{a,l}),$$

where $\chi(\widetilde{R}_{a,l})$ *is the orbifold Euler number of the orbifold* $\widetilde{R}_{a,l}$*, see Section 2.1. And so for any other contact form on* C*, homotopic (through contact forms) to* λ *there must be at least* ceil $|(\sum_{a,l}(-1)^{n+1}\chi(R_{a,l})|$ *Reeb orbits in class* $[o]$.

Here is one elementary example.

Example 2. Take (C,ξ) to be the unit cotangent bundle of a hyperbolic manifold. For a contact form λ induced by a given hyperbolic metric, the induced Reeb flow has a single Reeb orbit in each homotopy class. The same holds for any other contact form homotopic through contact forms to λ.

When we are in a Morse-Bott setting but the associated Cauchy-Riemann operators are not complex linear, the situation is more complicated. However given some mild additional assumptions on the contact form, Bourgeois [3] shows how to perturb λ is such a way that all periodic orbits are associated to critical points of Morse functions on the Reeb orbifolds, with explicitly determined Conley-Zehnder index, we may then in principle compute $\mathcal{E}(C, A_o)$ directly from that perturbed data, using Theorem 1.10 in the bounded case, and directly from the Definition 1.8 in the infinite case. We will not explicitly state this as a theorem for lack of truly interesting examples.

Theorem 1.14. *For the standard contact structure on* S^{2n+1}*,* $\mathcal{E}(C) = -\infty$*. In particular for any contact form homotopic through contact forms to the standard contact form there are Reeb orbits.*

The last assertion of course follows by contact homology techniques, but is very simple via our Gromov-Witten invariants. The above has a certain simple generalization to prequantization spaces. (By no means absolutely general.) Let C be the so called pre-quantization circle bundle associated to a Hermitian holomorphic line bundle L over a Kahler manifold (M,ω,j), with Chern connection ∇, whose curvature 2-form is $-2\pi i\omega$. Thus this is a Kahler prequantization space: (M,L,∇,ω,j). C is associated to (M,L,∇,ω,j) by taking the associated principal circle bundle, (which is just the "unit" circle sub-bundle of L) with connection form given by $i\lambda$ - the lie algebra valued connection 1-form determined by ∇, and λ is our contact form. In this case the Reeb flow is 2π-periodic and is just given by the natural right T action.

Theorem 1.15. *Let* (C,α) *be the prequantization circle bundle associated to* (M,L,∇,j) *as above, with* $\chi(M)\neq 0$*. Let* $[o]$ *be a class which is represented by a (non-constant) Reeb orbit of* α*. If* $[o]$ *is non-torsion then for any contact form* λ *homotopic (through contact forms) to* α*, there is a* R^λ*-Reeb orbit in class* $[o]$*. If* $[o]$ *is trivial or torsion and* M *admits a vector field with non-degenerate zero's whose indexes have positive index, then for any contact form* λ *homotopic (through contact forms) to* α *there is a* R^λ*-Reeb orbit in class* $[o]$*.*

Example 3. Take a Kahler submanifold $M\subset\mathbb{CP}^n$ which admits a perfect Morse function, or just a vector field with non-degenerate zeros with positive index, and any class $[o]$, which is represented by a Reeb orbit.

The theorem above can be obtained via (non-cylindrical) contact homology techniques see Borgeouis [3]. We shall prove this in Section 3.

Here is another is another application of more qualitative kind. I don't know if one can obtain this via contact homology. Although it appears that it can also be proved by the classical Fuller index theory (without any extensions), given Lemma 3.10. (Which also means that there are extensions of the theorem beyond Reeb vector fields.) Moreover it seems to be close in spirit to some of the work of Kerman [10], which uses Floer theoretic techniques.

Theorem 1.16. *Let* (C,ξ) *be a closed contact* $2n+1$*-fold,* $\lambda_1 = f\lambda$*,* $t\in[0,1]$*,* $f\geq 1$*, with* λ *non-degenerate. Let* $C=\max_C f-1$*. Suppose that there are no* R^λ*-Reeb orbits with period*

$$P \leq \text{period} \leq lnC\cdot P.$$

Then if

$$\#\mathcal{M}_1(J^\lambda, A)_{2\pi\cdot P} = N \neq 0$$

there are at least ceil $|N|$ *periodic orbits of* λ_1 *with period at most* $\ln C\cdot P$*. Moreover if* P *is the minimal period, and* $\ln C<2$*, then Reeb orbits of* λ_1 *with action at most* $\ln C\cdot P$ *are distinct and embedded.*

1.5. **Relative theory.** Most of the content of this paper has a relative analogue. We shall only sketch this out as the arguments are mostly identical. First a relative analogue of theorem 1.3, which for simplicity (to avoid extra difficulty with moduli spaces of Riemann surfaces with boundary) we state for annuli, which is what is relavant for the main example of the paper, and for relative "Fuller theory".

Theorem 1.17. *Suppose that X is a closed* l. c. s. m.*, and $L \subset X$ a closed Lagrangian submanifold. Let $\mathcal{M}_1(L, J, A)$ be the space of equivalence classes of J-holomorphic maps $(An, \partial An) \to (X, L)$ in relative class class A, where An is a Riemann surface with boundary, diffeomorphic to an annulus. Then $\mathcal{M}_1(L, J, A)$ has a metric completion*

$$\overline{\mathcal{M}}_1(L, J, A),$$

by Kontsevich stable maps. Moreover given $E > 0$, the subspace $\overline{\mathcal{M}}_1(L, J, A)_E \subset \overline{\mathcal{M}}_1(L, J, A)$ consisting of elements with L^2 energy $\leq E$ is compact.

For the relative analogue of our example Gromov-Witten theory of $C \times S^1$, we take a Lagrangian submanifold L of $C \times S^1$, of the form $Leg_1 \times S^1 \sqcup Leg_2 \times S^1$ for Leg_i closed Legendrian submanifolds of C. There are distinguished holomorphic annuli with boundary on L, which are constructed as follows. Recall that a Reeb chord γ of R^λ, going from Leg_1 to Leg_2, is a smooth map $\gamma : [0, 1] \to C$, $\gamma(0) \in Leg_1$, $\gamma(1) \in Leg_2$, $D_t\gamma(t) = cR^\lambda(\gamma(t))$, $c > 0$. Given a Reeb chord γ from Leg_1 to Leg_2, a *Reeb annulus* u_γ is the map

$$u_\gamma(r, \theta) = (\gamma(r), \theta),$$

using polar coordinates on the annulus. As with Reeb tori a Reeb annulus is J^λ-holomorphic for a uniquely determined holomorphic structure j on An. If

$$D_t\gamma(t) = c \cdot R^\lambda(\gamma(t)),$$

then

$$j(\frac{\partial}{\partial r}) = c\frac{\partial}{\partial \theta}.$$

Let A denote the class of such a curve. We then have an analogue of Proposition 1.4:

Proposition 1.18. *Let (C, λ) and $L \subset C$ be as above. The moduli space $\overline{\mathcal{M}}_1(L, J^\lambda, A)$ consists only of Reeb annuli. For a pair $\widetilde{J}(\lambda_1), \widetilde{J}(\lambda_2)$ of almost complex structures as above and $\{\widetilde{J}(\lambda_t)\}$ a smooth interpolating family, the moduli space $\overline{\mathcal{M}}_1(L, \{\widetilde{J}(\lambda_t)\}, A)$, has compact connected components.*

As the normal bundle N_u, $[u] \in \overline{\mathcal{M}}_1(J^\lambda, A)$, and its real subbundle over the boundary corresponding to L are both simultaneously trivial, by the Riemann-Roch theorem the associated Fredholm operator has index $0+1$, with $+1$ for the dimension of the moduli space of complex annuli. The formal dimension of $\overline{\mathcal{M}}_1(J^\lambda, A)$ is then 1 minus the dimension of the automorphism group (for smooth curves) which is 1, so is 0.

The regularity also works as with Reeb tori. Recall that an action a Reeb cord γ between Legendrian submanifolds Leg_0, Leg_1 of a contact $2n+1$-fold C is called non-degenerate if $\phi^\lambda_{a,*}(T_{\gamma(0)}Leg_0)$ intersects $T_{\gamma(a)}Leg_a$ transversally, where $\{\phi^\lambda_{t,*}\}$ is the linearized at γ, R^λ-Reeb flow. We say that λ is non-degenerate relative to Leg_0, Leg_1 if all the Reeb cords between Leg_i are non-degenerate.

Proposition 1.19. *Let (C, ξ) be a closed contact manifold. Suppose that λ is non-degenerate relative to closed Legendrians Leg_0, Leg_1 then the moduli space $\overline{\mathcal{M}}_1(L, J^\lambda, A)$ is regular.*

Remark 1.20. To naturally orient $\overline{\mathcal{M}}_1(L, J^\lambda, A)$, we do not need any additional assumptions on L, this is because for a curve $[u] \in \overline{\mathcal{M}}_1(L, J^\lambda, A)$, the normal bundle over the boundary of the curve and its real subbundle corresponding to L are simultaneously naturally trivial. We claim that the following version of Proposition 1.7 holds. Let (C, λ) be a closed contact $2n+1$-fold, λ non-degenerate with respect to closed Legendrians Leg_1, Leg_2 and γ an action a R^λ Reeb cord from Leg_1, to Leg_2 then there is natural determinant line bundle theoretic orientation of $[u_\gamma]$, which is identified with the natural orientation of the intersection $\phi^\lambda_{a,*}(T_{\gamma(0)}Leg_0) \cap T_{\gamma(a)}Leg_1$. We will not verify this here as it is not a priority.

We shall not pursue orientation issues for the annular moduli spaces, and content ourself with defining our invariants $\mod 2$ which is possible because there are no isotropy groups to deal with in this case.

Definition 1.21. *Let (C, ξ), be a contact $2n+1$-fold, λ non-degenerate with respect to $\{Leg_i\}$, with finitely many Reeb chords in class $[\gamma]$ going from Leg_1, to Leg_2, and $L = Leg_1 \times S^1 \sqcup Leg_2 \times S^1$ as before, A the class of Reeb annulus for a Reeb chord in class $[\gamma]$. We define*

$$\mathcal{E}(C \times S^1, L, A) = \#\overline{\mathcal{M}}_1(L, J^\lambda, A) \mod 2,$$

where $\#\overline{\mathcal{M}}_1(L, J^\lambda, A) \mod 2$ is the $\mod 2$ count of elements of the space.

Since we have avoided orientation issues we will not discuss the infinity type cases, although of course we claim that this can be done.

Lemma 1.22. *The invariant $\mathcal{E}(C \times S^1, L, A)$ is well defined, (when defined).*

The proof is analogous to the proof of Lemma 1.9. This together with Proposition 1.19, this readily implies the following.

Theorem 1.23. *Given a closed contact $2n+1$-fold C, with Leg_1, Leg_2, closed Legendrian submanifolds, and λ non-degenerate with respect to Leg_i, if there is an odd number of class $[\gamma]$, R^λ-Reeb chords between Leg_1, Leg_2, then for any λ' homotopic (through contact forms) to λ there is at least one class $[\gamma]$ R^λ-Reeb chord between Leg_1, Leg_2.*

In this generality, this may also be impossible to obtain via contact homology.

1.6. **Relative Fuller index.** Naturally since our invariants $\mathcal{E}(C \times S^1, A)$ are closely related to the Fuller index, the invariants $\mathcal{E}(C \times S^1, L, A)$ should be related to some kind of relative Fuller index, strangely this does not appear anywhere in literature. But there should be no essential difficulty in making such a construction.

2. Preliminaries

2.1. **Virtual Moduli cycles and Gromov-Witten invariants.** In applications here our moduli spaces (for fixed J) will be given by orbifolds with orbifold obstruction bundles! So we do not have to worry about working with Kuranishi atlases. We need the virtual moduli cycle theory only to tell us how we should count (especially elements with symmetry groups) and to tell us that these counts are invariant.

Given a closed oriented orbifold X, with an orbibundle E over X Fukaya-Ono [7] show how to construct using multi-sections its rational homology Euler class, which when X represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. (Note that the story of the Euler class is older than the work of Fukaya-Ono, and there is possibly prior work in this direction.) When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reason. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [12], which does not have the effectivity assumption, a similar use of branched manifolds appears in [4]. In the case of a non-effective orbibundle $E \to X$ McDuff [13], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [7]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of E, constructed by other authors. In other words there is a completely natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

Axiom 2.1. *Suppose that the moduli space of holomorphic curves is represented by a (non-effective) orbifold X with an orbifold obstruction bundle E. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [6], although not quiet immediately. This is also communicated to me by Kaoru Ono. When X is 0-dimensional this does follow immediately by the construction in [7], taking any effective Kuranishi neighborhood at the isolated points of X, (this suffices for the majority of our paper.) We shall assume that our black box is represented by [6], since this work is effectively complete, for all genus curves.

As a special case most relevant to us here, suppose we have a moduli space of genus 1 curves X, which is regular with Fredholm index 0. Then its underlying space is a collection of points. However as some curves are multiply covered, and so have symmetry groups, we must treat this is a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{[\Gamma([u])]}$, where $[\Gamma([u])]$ is the order of the symmetry group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [12, Section 5]. In the setup of Fukaya-Ono [7] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of X.

2.2. **Virtual fundamental classes and chains.** First we should quickly point out that the construction of Kuranishi structures, or any similar construction like Kuranishi atlases of McDuff-Wehrheim obviously extends to moduli spaces of J-holomorphic curves in a non-symplectic manifold, provided that the moduli space, that is the zero set of the non linear Cauchy-Riemann section, is compact (modulo reparametrization group), (possibly after adding in Kontsevich stable maps). In the main example of this paper compactness holds for connected components and we work on the level of these components.

Given the above, we sometimes use notation like

$$\mathcal{C}^{vircycle}, \text{ or } \mathcal{C}^{virchain}$$

for $\mathcal{C}$ a connected component (which are compact) of $\overline{\mathcal{M}}_1(J^\lambda, A)$, or respectively for connected components of cobordism moduli space $\overline{\mathcal{M}}_1(\{J^{\lambda_t}\}, A)$, (which are also compact). By this we mean the following. When X comes with a Kuranishi structure $\mathcal{K}$ after some intermediate steps (construction of a good coordinate system, etc.) we choose a perturbed (multi)-section $s + \nu$ and a triangulation of its zero set. Then appropriately choosing rational weights as in [7], we get a virtual fundamental cycle, $X^{vircycle}_{choices,\mathcal{K}}$, whose homology class $[X]^{vir}_{\mathcal{K}}$ is independent of the choices. Likewise when X is a Kuranishi cobordism, we get a virtual fundamental chain $X^{virchain}_{choices,\mathcal{K}}$.

When X represents (a component of) the moduli space of holomorphic curves $\mathcal{K}$ is natural up to concordance, and we choose to omit $choices, \mathcal{K}$ from the notation thinking of them implicitly.

3. Proofs

Proof of Theorem 1.1. Suppose that we have a u as in the hypothesis, then du is non-singular outside a set of measure 0: $\Theta \subset \Sigma$, with respect to an auxiliary Riemannian metric on Σ.

Lemma 3.1. *For u as above the magnitude of the mean curvature $\boldsymbol{H}$ of the image of u is bounded by a universal constant $C(\omega, J) > 0$, outside Θ.*

Proof. We may fix a finite cover of M by charts $\phi_i : U_i \subset \mathbb{R}^{2n} \to M$, $\phi_i^*\omega = e^{f_i}\omega_0$ with ω_0 symplectic, with U_i contractible. Then $\phi_i^{-1} \circ u$ is a $\phi_i^* J$-holomorphic map defined on $u^{-1}(\phi_i(U_i))$ into $\mathbb{R}^{2n}$, and $\phi_i^* J$ is compatible with $f_i\omega_0$ and hence with ω_0. Fix an (infinite) full measure cover by closed disk domains $\{V_{i_j}\}$ of $\Sigma - \Theta$, with each $V_{i_j} \subset u^{-1}(\phi_i(U_i))$ for some i. Then $\phi_i^{-1} \circ u|_{V_{i_j}}$ is an immersed $\phi_i^* J$-holomorphic curve in $\mathbb{R}^{2n}$. As this complex structure is compatible with the symplectic form ω it classically follows that the image D_{i_j} of $\phi_i^{-1} \circ u|_{V_{i_j}}$ is a minimal surface and hence has mean curvature 0 with respect to $(\omega_0, \phi_i^* J)$. Since the cover $\{U_i\}$ is finite and since the C^∞ norm of the functions e^{f_i} is obviously bounded, it follows that the magnitude of the mean curvature of the surface D_{i_j} with respect to the metric induced by $(e^{f_i}\omega_0, \phi_i^* J)$ is bounded by some universal constant $C(\omega, J)$. Consequently the same holds for image $u|_{V_{i_j}}$ with respect to g_J from which the result follows. $\square$

Next we shall need the following inequality relating diameter and mean curvature which readily follows by a Theorem of Topping [17].

Lemma 3.2. *For a compact Riemannian manifold (M, g), and $f : \Sigma \to M$ a closed surface in M, which is non-singular on a set of full-measure (with respect to an auxiliary metric g' on Σ):*

$$\operatorname{diam}_g(\operatorname{image} f) \leq C(M, g) \int_\Sigma |\boldsymbol{H}_{M,g} \circ f| dvol_g,$$

where

$$\operatorname{diam}_g(\operatorname{image} f) := \max_{x,y \in \Sigma} dist_{\Sigma,g}(f(x), f(y)),$$

$dist_{\Sigma,g}$ is the induced metric from the ambient space, and $dvol_g$ is the measure on Σ induced by g.

Proof. The theorem of Topping is for closed immersed submanifolds of R^n, and for surfaces $\Sigma \subset \mathbb{R}^n$ says that:

$$\operatorname{diam}(\Sigma) \leq C(n) \int_\Sigma |\mathbf{H}_{R^n}| dvol_\Sigma,$$

where diam is the intrinsic diameter: $\max_{x,y\in\Sigma} dist_{\Sigma,g}(x, y)$, and where $dvol_\Sigma$ is the measure induced from the standard metric on $\mathbb{R}^n$. To obtain the version stated for a more general but compact (M, g), and non-immersed surfaces, pick an isometric Nash embedding N of (M, g) into $\mathbb{R}^n$, where n is large enough. Take a small perturbation $(N \circ f)'$ of $N \circ f$ so that $(N \circ f)'$ is an immersion (or even embedding) into $\mathbb{R}^n$. Since Σ is closed we get by Topping's theorem:

$$\operatorname{diam}(\operatorname{image}(N \circ f)') \leq C(n) \int_\Sigma |\mathbf{H}_{\mathbb{R}^n} \circ (N \circ f)'| \, dvol_\Sigma.$$

Note that under hypotheses of the theorem the measures $dvol_g$ and $dvol_{g'}$ on Σ are equivalent, specifically because f is non-singular on a set of full $dvol_{g'}$ measure. Since M is compact the function $z \mapsto |\mathbf{H}_{R^n}(N \circ f(z))|$ on Σ is bounded on a set of full $dvol_{g'}$ measure, and hence $dvol_g$ measure, from above by the function $z \mapsto C'(N)|\mathbf{H}_{M,g}(f(z))|$ on Σ for some $C'(N) >> 0$, independent of f. This $C'(N)$ is just an upper bound for the function λ on M, s.t. $\lambda(m) \cdot |\mathbf{H}_{\mathbb{R}^n}(N(m))| = |\mathbf{H}_g(m)|$, $m \in M$.

So we get:

$$\operatorname{diam}_g(\operatorname{image} f) = \operatorname{diam}(\operatorname{image} Nf) \approx \operatorname{diam}(\operatorname{image} Nf') \leq C(n) \int_\Sigma |\mathbf{H}_{\mathbb{R}^n}((N \circ f)'(z))| \, dvol_\Sigma$$

$$\approx C(n) \int_\Sigma |\mathbf{H}_{\mathbb{R}^n}(N \circ f(z))| \, dvol_g \leq C'(N) \cdot C(n) \int_\Sigma |\mathbf{H}_{M,g}(f(z))| \, dvol_g,$$

where the approximate equalities $\approx$ become equalities in the limit that $d_{C^\infty}(N \circ f, (N \circ f)') \mapsto 0$. For the previous assertion to hold for the second approximate equality, we use that f is non-singular on a set of full measure, and then the assertion is completely elementary. So we get the required inequality. □

Finally let ϵ be the Lebesgue covering number of $\{U_i\}$ with respect to the metric g. Combining Lemma 3.1 and Theorem 3.2 we get that for u as in the hypothesis if $\operatorname{area}(u) < \hbar$ than $\operatorname{diam}(u) < \epsilon$, for some $\hbar$ independent of u. Consequently the image of u is contained in some U_i, and so $\phi_i^{-1} \circ u$ is a $\phi_i^* J$-holomorphic map of a sphere into the almost Kahler contractible manifold $(U_i, \omega_0, \phi_i^* J)$ and so must be constant. □

Proof of Theorem 1.3. (Outline, as the argument is standard.) Suppose that we have a sequence u^k of J-holomorphic maps with L^2-energy $\leq E$. By [15, 4.1.1], a sequence u^k of J-holomorphic curves has a convergent subsequence if $sup_k ||du^k||_{L^\infty} < \infty$. On the other hand when this condition does not hold rescaling argument tells us that a holomorphic sphere bubbles off. The quantization Theorem 1.1, then tells us these bubbles have some minimal energy, so if the total energy is capped by E, only finitely many bubbles may appear, so that a subsequence of u^k must converge in the Gromov topology to a Kontsevich stable map. □

Proof of Proposition 1.4. T acts on the moduli space by post-composition of class representative stable maps with the J^λ preserving action of T on $C \times S^1$.

Lemma 3.3. *The entire moduli space $\overline{\mathcal{M}}_1(\{J^\lambda_t\}, A)$, is fixed by the T action and consists of Reeb tori.*

Proof. Suppose we a have a smooth (non-nodal curve) $[u] \in \overline{\mathcal{M}}_1(\{J^\lambda_t\}, A)$. Pick an identification of the domain Σ of u with a Riemann surface (T^2, j), T^2 the standard torus. We shall use throughout coordinates (θ_1, θ_2) on T^2 $\theta_1, \theta_2 \in S^1$, with S^1 unit complex numbers. Then if $u \in [u]$ by assumption on the class $A = [u]$, $\theta \mapsto pr \circ u(\{\theta^1_0\} \times \{\theta\})$, is a degree 1 curve, where $pr : C \times S^1 \to C$ is the projection. If $u \in [u]$ is a representative then we claim that $(pr_C \circ u)_*$, has rank everywhere ≤ 1. Suppose otherwise than it is immediate by construction of J^λ, that $\int_{T^2} u^* d\lambda > 0$, but $d\lambda$ is exact so that that this is impossible. Next observe that when the rank of $(pr_C \circ u)_*$ is 1, its image is in the Reeb line sub-bundle of TC, for otherwise the image has a contact component, but this is J^λ invariant and so again we get that $\int_{T^2} u^* d\lambda > 0$. We now show that the image of $pr_C \circ u$ is in fact the image of some Reeb orbit.

By assumption $pr_{S^1} \circ u$ is onto S^1 and so by the Sard theorem we have a regular value θ_0, so that $S_0 = u^{-1} \circ pr^{-1}_{S^1}(\theta_0)$ is an embedded circle in T^2. Now $d(pr_{S^1} \circ u)$ is surjective along $T(T^2)|_{S_0}$, which means, since u is J^λ-holomorphic that $pr_C \circ u|_{S_0}$ has non-vanishing differential. From this and the discussion above it follows that image of $pr_C \circ u$ is the image of some embedded Reeb orbit o_u. Consequently the image of u is contained in the image of the Reeb torus of o_u, and so u is a itself a Reeb torus map for some o covering o_u, and so is T-invariant.

The statement of the lemma follows when $[u]$ is non-nodal. When $[u]$ is a nodal curve, the previous argument applied to the principal (that is non-spherical) component shows that it must be smooth (non-nodal). On the other hand non-constant holomorphic spheres are impossible also by the previous argument. So there are no nodal elements in $\overline{\mathcal{M}}_1(J^\lambda, A)$ which completes the argument. ∎

To prove the claim with regards to cobordisms, it suffices to prove Lemma 1.5. Since then the connected components of the S^1-quotient of S by the reparametrization S^1 action are also compact, and by the previous discussion these are identified with connected components of $\overline{\mathcal{M}}_1(\{J^{\lambda_t}\}, A)$.

Proof of Lemma 1.5. Given the space S of pairs (o, t), o a Reeb orbit o for R^{λ_t}, we show that the function

$$\Lambda : S \to \mathbb{R}, \quad \Lambda([o], t) = \langle [\lambda_t], [o] \rangle$$

is bounded on connected components, where $\langle , \rangle$ denotes the integration pairing. As Λ is a proper function on S applying Azrela-Ascoli theorem, this will readily imply our Lemma.

Suppose otherwise that Λ is not bounded on connected components of S. Then there is a continuous map $p : [0, 1) \to S$ with $\Lambda(p(\tau)) - \Lambda(p(0))$ tending to ∞ as $\tau \mapsto 1$ such that given the smooth projection

$$\pi^0 : S \to [0, 1], \quad \pi^0(o, t) = t,$$

we have

$$||\pi^0_*(D_\tau p)|| \leq 1, \quad \forall \tau \in [0, 1). \tag{3.4}$$

Uniformly approximate p in C^∞ topology by a smooth map $p' : [0, 1) \to S$. Then by the chain rule

$$D_\tau|_{\tau_0} (\Lambda \circ p'(\tau)) = D_\tau|_{\tau_0}(\Lambda \circ p'(\tau_0)) + D(\lambda_{\pi^0(p'(\tau_0))})(\pi^0_* D_\tau|_{\tau_0} p'),$$

where $D(\lambda_{\pi^0(p'(\tau_0))})$ denotes the differential of the functional of integration of $\lambda_{\pi^0(p'(\tau_0))}$. The magnitude of the second term can be bounded by some ϵ by criticality of the λ functional on Reeb orbits, by assumption that p' uniformly approximates p, and by (3.4), while the first term can be bounded from above by by $Const \cdot \lambda_{\tau_0}(p'(\tau_0))$, for a constant $Const'$ independent of t since $[0, 1]$ is compact. Explicitly if $\lambda_t = f_t \lambda$, $f_t > 0$ then by direct calculation

$$D_\tau|_{\tau_0} (\Lambda \circ p'(\tau_0)) \leq Const \cdot \int_{p'(\tau_0)} \lambda \tag{3.5}$$

where

$$Const = \max_{t \in [0,1], C} \frac{df_t}{dt}.$$

On the other hand

$$\int_{p'(\tau_0)} \lambda \leq K\Lambda(p'(\tau_0)),$$

where $K = \max_{C,t} f_t$, if $\max_{C,t} f_t \geq 1$, $K = 1$ otherwise. Consequently:

$$D_\tau|_{\tau_0} (\Lambda \circ p'(\tau)) \leq Const \cdot K \cdot \Lambda \circ p'(\tau_0) + \epsilon, \tag{3.6}$$

for all $\tau_0 \in [0, 1)$. But this contradicts the hypothesis that $\Lambda(p(\tau)) - \Lambda(p(0))$ can be arbitrarily large. (The growth is sub-exponential). □

□

Proof of Proposition 1.6. We shall give two proofs as they are of independent interest. The first only applies in dimension 3, but has the advantage of being very explicit. The second is for general contact manifolds (l. c. s. m. of the form $C \times S^1$). We have previously shown that all $[u, j] \in \overline{\mathcal{M}}_1(J^\lambda, A)$, are represented by smooth immersed curves, (covering maps of Reeb tori) and these are fixed by the T action. Since each u is immersed we may naturally get a T-invariant splitting $u^*T(C \times S^1) \simeq N_u \times T(T^2)$, using g_J metric, where N_u denotes the pull-back normal bundle. The full associated real linear Cauchy-Riemann operator takes the form:

$$D^J_u : \Omega^0(N_u \oplus T(T^2)) \oplus T_j M_1 \to \Omega^{0,1}(T(T^2), N_u \oplus T(T^2)). \tag{3.7}$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N_u \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N_u) \oplus \Omega^0(T(T^2)) \to \Omega^{0,1}(T(T^2), N_u) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_1 \to \Omega^{0,1}(T(T^2)),$$

is surjective by classical algebraic geometry. It follows that D^J_u will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N_u) \to \Omega^{0,1}(N_u),$$

has no kernel.

Note that N_u is T-equivariantly trivial by geometry and moreover the T-action preserves D also by geometry. For the following we need that dimension of C is 3. A pair of elements $\xi_1, \xi_2 \in \ker D$ either coincide or are disjoint. For if they intersect $\xi_1 - \xi_2 \in \ker D$ vanishes at some $z \in T^2$, and by Aronsajn's unique continuation theorem it follows that any such zero must contribute positively to the self intersection number of the zero section, unless $\xi_1 - \xi_2$ vanishes identically c.f. for instance Taubes [16, Section 5]. In other words this is the "positivity of intersections" argument. It follows from this that the linear span of elements of $\ker D$, determines a sub-bundle $Ker \subset N_u$ of dimension $d := \dim \ker D$. Since Ker is spanned by $\ker D$ which is T-invariant Ker is T-invariant. Trivializing Ker so that its constant sections correspond to elements of $\ker D$, we see that the induced action of T on $\mathbb{R}^d \times T^2$ must be of the form

$$\theta \cdot (v, z) = (a(\theta)v, \theta \cdot T), \tag{3.8}$$

where $a(\theta) \in End(\mathbb{R}^d)$, for all θ, as by construction the induced action on $\mathbb{R}^d \times T^2$ must take constant sections to constant sections. We claim that $a(\theta) = id$ for all θ. To see this note that by geometry there is clearly a trivialization of N_u with respect to which T acts as in (3.8) with $a(\theta) = id$ for all θ. Since Ker is a T-invariant sub-bundle of N_u, it follows that in this trivialization $Ker \subset \mathbb{R}^2 \times T^2$, is of the form $\mathbb{R}^d \times T^2 \subset \mathbb{R}^2 \times T^2$. So if $a(\theta) \neq id$, for all θ, there is a T-equivariant endomorphism of $\mathbb{R}^d \times T^2$ with respect to a pair of T-actions in the form of (3.8) one of which is trivial in the $\mathbb{R}^d$ variable, which implies the same for the other. So we obtain in the contact 3-fold case:

Lemma 3.9. *Any element μ of $\ker D$, $D : \Omega^0(N_u) \to \Omega^{0,1}(T^2, N_u)$ must be T-invariant.*

Suppose that $\ker D \neq 0$, we use this to obtain an eigenvector of the time-1 linearized return map of the Reeb flow at the orbit o_u, which will give a contradiction. As μ as T-invariant and as the g_J exponential map from N_u into $C \times S^1$ is T-invariant, we obtain a smooth T-invariant embedded submanifold

$$T^2 \times [0, \epsilon) \to C \times S^1,$$

containing image u, defined by $(z, t) \mapsto (exp(t\mu(z)))$. And we have a vector field κ on this submanifold S_u, given by the pushforward of the rotational vector field on $T^2 \times [0, \epsilon]$ which in coordinates is $(\frac{\partial}{\partial \theta_1}, 0, 0)$, for coordinates (θ_1, θ_2, t). The flow $\phi^\kappa(\tau)$ of κ induces the linearized flow $\phi^\kappa_*(\tau)$ on the sub-bundle $\mathbb{R} \cdot \mu \subset N_u$, which at $\mu(z)$ is just $\mu_*(\frac{\partial}{\partial \theta_1}(z))$. On the other hand as μ is in $\ker D$, it follows by definition of D, and T-invariance of S_u that κ C^1 converges, as we approach image $u \subset S_u$, to $J^\lambda(\frac{\partial}{\partial \theta})$, which is the Reeb vector field R^λ of λ. Consequently the linearization of the Reeb flow preserves $\mathbb{R} \cdot \mu \subset N_u$ and coincides with the linearization of the κ flow. In particular $\mu|_{S^1 \times \{\theta_2\}}$ must be an integral curve of the linearized Reeb flow, and since it is closed this is a contradiction to o_u being non-degenerate.

We now give a proof which works in all dimensions. The bundle N_u is of course symplectic with symplectic form on the fibers given by restriction to the fibers of the pullback by u, of the form $d\lambda$, together with J^λ this gives a Hermitian structure on N_u. We have a linear symplectic connection A on N_u, which over the slices $S^1 \times \{\theta_2'\} \subset T^2$ is induced by the pullback by u of the linearized R^λ Reeb flow. Specifically the A-transport map from $N|_{(\theta_1', \theta_2')}$ to $N|_{(\theta_1'', \theta_2')}$ over $[\theta_1', \theta_2''] \times \{\theta_2'\} \subset T^2$, $0 \leq \theta_1' \leq \theta_2'' \leq 2\pi$ is given by

$$u_*|_{N|_{(\theta_1'', \theta_2')}} \circ \phi^R_{mult \cdot (\theta_1'' - \theta_1')} \circ u_*|_{N|_{(\theta_1', \theta_2')}},$$

where $mult$ is the multiplicity of o and where $\phi^R_{mult \cdot (\theta_1'' - \theta_1')}$ is the time $mult \cdot (\theta_1'' - \theta_1')$ map for the R^λ Reeb flow.

The connection A is defined to be trivial in the θ_2 direction, where trivial means that the parallel transport maps are the id maps over θ_2 rays. In particular the curvature R_A of this connection vanishes. The connection A determines a real linear CR operator on N_u in the standard way (take the complex anti-linear part of the vertical differential of a section). It is easy to verify that this operator is exactly D. We have a differential 2-form Ω on the N_u which in the fibers of N_u is just the fiber symplectic form and which is defined to vanish on the horizontal distribution. This 2-form is closed, the conceptual way to see this is to observe that this is just the coupling 2-form for our symplectic connection A, see for instance [14] for the basics on coupling forms, however we can also check that it is closed explicitly by peaking good coordinates. (Use that R_A vanishes to locally integrate the horizontal distribution.)

Clearly Ω must vanish on the 0-section since it is a A-flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then Ω vanishes on μ. But then since $\mu \in \ker D$, and so its vertical differential is complex linear, it must follow that the vertical differential vanishes, since $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert} N_u$ and so otherwise we would have $\int_\mu \Omega > 0$. So μ is A-flat, in particular the restriction of μ over all slices $S^1 \times \{\theta_2'\}$ is identified with a period $period(o)$ orbit of the linearized at o R^λ Reeb flow, which does not depend on θ_2' as A is trivial in the θ_2 variable. So the kernel of D is identified with the vector space of period $period(o)$ orbits of the linearized at o R^λ Reeb flow, as needed. □

Proof of Proposition 1.7. Abbreviate u_o by u. Fix a trivialization ϕ of N_u induced by a trivialization of the contact distribution ξ along o in the obvious sense: N_u is the pullback of ξ along the composition

$$T^2 \to S^1 \xrightarrow{o} C.$$

Then the pullback A' of A to $T^2 \times \mathbb{R}^{2n}$ is a connection whose holonomy path p along $S^1 \times \{\theta_2\}$ is θ independent and so that the parallel transport path of A' along $\{\theta_1\} \times S^1$ is constant, that is A' is trivial in the θ_2 variable. We shall call such a connection A' on $T^2 \times \mathbb{R}^{2n}$ *induced by* p. The holonomy path p is a smooth path $p : [0, 1] \to \mathrm{Symp}(\mathbb{R}^{2n})$, starting at 1, and by non-degeneracy assumption on o, the end point map $p(1)$ has no 1-eigenvalues. Let $p'' : [0, 1] \to \mathrm{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, s.t. p'' has only simple crossings with the

Maslov cycle. Let p' be the concatenation of p and p'', since p' is homotopic relative end points to a unitary geodesic path h starting at id, having regular crossings

$$CZ(p') - \Gamma(p', 0) = CZ(p'') - n \mod 2 = 0,$$

as the number of negative, positive eigenvalues is even at each regular crossing of h by unitarity. Consequently $CZ(p'') \mod 2 = CZ(p') - n \mod 2$ by additivity of the Conley-Zehnder index. Let us then define a free homotopy $\{p_t\}$ of p to p', p_t is the concatenation of p with $p''|_{[0,t]}$, reparametrized to have domain $[0, 1]$ at each moment t. This determines a homotopy $\{A'_t\}$ of connections induced by $\{p_t\}$. The CR operator D_t determined by each A'_t is surjective except at some finite collection of times $t_i \in (0, 1)$, $i \in N$ determined by the crossing times of p'' with the Maslov cycle, and the dimension of the kernel of D_{t_i} is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of p'' are simple. The operator D_1 is not complex linear so we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\widetilde{D}_t\}$ induced by the homotopy $\{\widetilde{A}_t\}$ of A'_1 to a unitary connection $\widetilde{A}_1$, where the homotopy $\{\widetilde{A}_t\}$, is through connections induced by paths $\{\widetilde{p}_t\}$, giving a homotopy of $p' = \widetilde{p}_0$ to a unitary path $\widetilde{p}_1$. Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\widetilde{D}_t\}$. By construction in the second half of the homotopy $\{D'_t\}$, D'_t is surjective. And D'_1 is induced by a unitary connection, since it is induced by unitary path $\widetilde{p}_1$. Consequently D'_1 is complex linear. By the above construction, for the homotopy $\{D'_t\}$, D'_t is surjective except for N times in $(0, 1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p')-n},$$

which was what to be proved. □

Proof of Theorem 1.10. By propositions 1.6, 1.7, if λ is non-degenerate of bounded type the moduli space $\overline{\mathcal{M}}(J^\lambda, A)$, is regular, consists only of Reeb tori $[u_o]$, with orientation of $[u_o]$ given by $-1^{CZ(o)-n}$. If o has multiplicity k, then $[u_o]$ has a symmetry group of order k, which is the isotropy group of $[u_o]$ in the orbifold $\overline{\mathcal{M}}(J^\lambda, A)$. Consequently the contribution to the orbifold Euler number of $\overline{\mathcal{M}}(J^\lambda, A)$ from $[u_o]$ is $-1^{CZ(o)-n}/k$. □

Proof of Lemma 1.9. If λ_0, λ_1 are both of bounded type then $\overline{\mathcal{M}}_1(J^{\lambda_0} := J^0, A)$, $\overline{\mathcal{M}}_1(J^{\lambda_1} := J^1, A)$ have Kuranishi structures and are Kuranishi cobordant by $\overline{\mathcal{M}}_1(\{J^t\}, A)$ which in this case is a compact moduli space as readily follows by Proposition 1.4.

Let us prove the infinite type case when λ_0, λ_1 are regular, as the general case is logically the same. For a given energy level $E > 0$, we define $\overline{\mathcal{M}}_1(\{J^t\}, A)_E^{vircycle}$ as follows. Take the union $\overline{\mathcal{M}}_1(\{J^t\}, A)_E$ of all the connected components of the moduli space $\overline{\mathcal{M}}_1(\{J^t\}, A)$, which contain the elements of $\overline{\mathcal{M}}_1(J^0, A)_E$, and of $\overline{\mathcal{M}}_1(J^1, A)_E$, with these just being energy E sublevel sets. These connected components must be in the $(E + C_E)$ energy sublevel set of $\overline{\mathcal{M}}_1(\{J^t\}, A)$, for $C_E > 0$ depending on E, by Proposition 1.4. Clearly $\overline{\mathcal{M}}_1(\{J^t\}, A) = \bigcup_E \overline{\mathcal{M}}_1(\{J^t\}, A)_E$ and each $\overline{\mathcal{M}}_1(\{J^t\}, A)_E$ is an open and closed subspace of $\overline{\mathcal{M}}_1(\{J^t\}, A)$, moreover each $\overline{\mathcal{M}}_1(\{J^t\}, A)_E$ is compact and so has a natural induced $d = 1$ Kuranishi structure with boundary (that is a Kuranishi cobordism) and so with respect to particular abstract perturbation data, there is an induced singular 1-chain $\overline{\mathcal{M}}_1(\{J^t\}, A)_E^{virchain}$ whose boundary corresponding to elements with energy $\leq E$ is identified with

$$-\overline{\mathcal{M}}_1(J^0, A)_E + \overline{\mathcal{M}}_1(J^1, A)_E,$$

where the latter is understood as a (in this case) canonical singular 0-chain, representing the homology Euler class of the non-effective 0-dimensinal orbifold $\overline{\mathcal{M}}_1(J^0, A)_E^{op} \sqcup \overline{\mathcal{M}}_1(J^1, A)_E$, see Section 2.1. Suppose that there is a λ_0 of infinite type (say positive) and λ_1 of finite type. Take E sufficiently large so that

$$\overline{\mathcal{M}}_1(J^1, A)_E = \overline{\mathcal{M}}_1(J^1, A),$$

and so that there are no negatively signed elements of $\overline{\mathcal{M}}_1(J^0, A)$, with energy larger than E. Let $\overline{\mathcal{M}}_1(J^0, A)_{E,E+E_C}$ denote the subspace of elements of $[u] \in \overline{\mathcal{M}}_1(J^0, A)$ with $E \leq \text{energy}([u]) \leq E+E_C$.

Then

$$\partial \overline{\mathcal{M}}_1(\{J^t\}, A)^{virchain}_E = -c_{E,E_C} - \overline{\mathcal{M}}_1(J^0, A)_E + \overline{\mathcal{M}}_1(J^1, A),$$

where by the positivity assumption c_{E,E_C} is a 0-chain with positive coefficients, corresponding to the canonical representing chain for the orbifold Euler class of some suborbifold of $\overline{\mathcal{M}}_1(J^0, A)_{E,E+C_E}$. We have

$$\int_{\partial \overline{\mathcal{M}}_1(\{J^t\}, A)^{virchain}_E} 1 = 0,$$

but this (together with the positive infinite type assumption) readily implies that we may take an $E' > E$ sufficiently large so that in addition to conditions on E above we have

$$\int_{\overline{\mathcal{M}}_1(J^0, A)_{E'}} 1 > \int_{\overline{\mathcal{M}}_1(J^1, A)} 1.$$

Then we still have

$$\partial \overline{\mathcal{M}}_1(\{J^t\}, A)^{virchain}_{E'} = -c_{E',E'_C} - \overline{\mathcal{M}}_1(J^0, A)_{E'} + \overline{\mathcal{M}}_1(J^1, A),$$

where c_{E',E'_C} is as before. But then we get

$$\int_{\partial \overline{\mathcal{M}}_1(\{J^t\}, A)^{virchain}_{E'}} 1 < 0,$$

which is impossible, and so we obtain a contradiction.

The case of λ_0 of infinite positive type and λ_1 of infinite negative type is similar.

□

Proof of Theorem 1.16.

Lemma 3.10. *Suppose that $\lambda_t = (1-t)\lambda + tf\lambda$, $f \geq 1$, and $\{o_t\}$, $t \in [0,1]$ a continuous family with o_t a λ_t-Reeb orbit. Then:*

$$\text{period}(o_0) \leq \text{period}(o_1) \leq e^{(\max_C f - 1)} o_0.$$

Proof. In the argument of the proof of Proposition 1.4, in (3.5) we have

$$0 \leq D_\tau|_{\tau_0} (\Lambda \circ \widetilde{p}'(\tau_0)) \leq Const \cdot \int_{\widetilde{p}'(\tau_0)} \lambda \leq (\max_C f - 1) \cdot \Lambda(\widetilde{p}'(\tau_0)),$$

since $f \geq 1$. Since ϵ can be made arbitrarily small, we get

$$\Lambda(\widetilde{p}'(0)) \leq \Lambda \circ \widetilde{p}'(1) \leq \Lambda(\widetilde{p}'(0)) \cdot e^{\max_C f - 1}.$$

□

Given the above lemma it follows that the space $\overline{\mathcal{M}}_1(\{J^{\lambda_t}\}, A)_{2\pi P e^{Const}}$ (defined in the proof of Lemma 1.9 just above) is contained entirely in the $2\pi P e^{Const}$-sublevel set, for energy, from which the first part of the theorem immediately follows.

To prove the second part note that the minimal period grows in t by Lemma 3.10. Consequently R^{λ_1}-Reeb orbits with action bounded from above by Pe^{Const} cannot be multiply covered for $Const < \ln 2$, from which the conclusion follows. □

Proof of Theorem 1.13. We need to show:

$$\mathcal{E}(C, A_o) = \sum_{a,l} (-1) \chi(\widetilde{R}_{a,l}),$$

Lemma 3.11. *The orbifold obstruction bundle to the moduli (orbifold) space $\widetilde{R}_{a,l}$ is naturally isomorphic to the orbifold cotangent bundle of $\widetilde{R}_{a,l}$.*

Proof. Given $[u] \in \widetilde{R}_{a,l}$, by Serre duality the fiber of the obstruction bundle at u is $H^1(N_u) \simeq H^0(N_u^* \otimes K_u)$, where K_u is the canonical line bundle of the domain curve. Now K_u is holomorphically trivial, and N_u is holomorphically trivial as it is a line bundle with non-trivial $H^0(N_u) = \ker D = T_{[u]} R_a$ (by the Proposition 1.6) and vanishing degree. It follows that $H^0(N_u^* \otimes K_u) \simeq (H^0(N_u))^* = (\ker D)^*$. (the isomorphism is natural once one chooses an isomorphism $H^0(K_u) \simeq \mathbb{C}$, however this isomorphism is also natural as domain of u is biholomorphic to a torus T^2_a as described in the proof Lemma 3.12, for which there is a canonical identification $H^0(T^2_a) = \mathbb{C}$. Meanwhile the map $H^0(K_u) \to H^0(T^2_a)$ is obviously independent of the choice of the biholomorphism. (Any two are related by an automorphism of T^2_a, which act trivially on $H^0(T^2_a)$).

□

So the contribution to $\mathcal{E}(C, A_o)$ from $\widetilde{R}_{a,l}$ is the orbifold Euler number of the orbifold cotangent bundle of $\widetilde{R}_a$ or $-\chi(\widetilde{R}_a)$.

A note on orientations. As usual we orient using Quillen's determinant line bundle over the space of real linear Cauchy-Riemann operators with fiber $\mathrm{Det}(\ker D)^* \otimes \mathrm{Det}(\operatorname{coker} D)$, with Det denoting top exterior power. Sometimes one takes the dual on the second factor instead, the version above is Quillen's original definition which is rather intuitive, although it has a side effect of orienting the cotangent bundle when coker vanishes instead of the tangent bundle, but this is of no consequence. For a complex linear D as in the geometric situation above, an orientation preserving (for the canonical complex orientations) isomorphism $\ker D \to \operatorname{coker} D$ gives a "positive" orientation element of the determinant line. In particular for a transverse (multi)-section of the orbifold cotangent bundle of R_a, an intersection point p contributes positively if the associated vertical tangent map $T_p R_{a,l} \to (T_p R_{a,l})^*$ preserves the complex orientation.

Lemma 3.12. *On connected components the forgetful map*

$$fog : \widetilde{R}_a \to M_1$$

is constant. Moreover the image is away from orbifold points of M_1 unless $\frac{a}{2\pi} = 1$.

Proof. Each Reeb torus corresponding to action a Reeb orbit o is clearly (by geometry of the Reeb tori) bi-holomorphic to the quotient T^2_a, identifying opposite sides, of the rectangular domain in $\mathbb{C}$ with sides of length (for the standard Kahler metric) 1 and $\frac{a}{2\pi}$. So the complex structure on the domain curve for an element of $\widetilde{R}_a$ is completely determined by the action of the underlying Reeb orbit. On the other hand the action is clearly constant on the connected component R_a. The last assertion clearly follows since T^2_a has no symmetries unless $\frac{a}{2\pi} = 1$. □

□

Proof of Theorem 1.14. The standard contact form is Morse-Bott with the corresponding components of the moduli space $\widetilde{R}_{k2\pi}$, $k \geq 1$ being orbifold quotients by the trivial $\mathbb{Z}_k$ action of $\mathbb{CP}^n$. In this case the associated Cauchy-Riemann operators are complex linear and the obstruction bundle for each component is just the (orbifold) cotangent bundle of $\mathbb{CP}^n$, by the proof of Corollary 1.13, so the contribution from each component $\widetilde{R}_{k2\pi}$, $k \geq 1$ is $-\frac{1}{k}\chi(\mathbb{CP}^n)$. We may then construct abstract perturbations on each component $\widetilde{R}_{k2\pi}$ corresponding to a section of the cotangent bundle of $\mathbb{CP}^n$ which has isolated non-degenerate zeros with negative index, so that $\widetilde{R}^{vircycle}_{2\pi}$ is a 0-dimensional oriented weighted branched manifold with negatively signed elements, and so λ is of negative infinite type. □

Proof of Theorem 1.15. Given our holomorphic vector bundle $L \to M$, we get a holomorphic torus bundle as follows. Take $L - 0$ that is L minus the 0-section. This is a holomorphic $\mathbb{C} - 0$ bundle with structure group $\mathbb{C}^\times$. There is a natural holomorphic $\mathbb{Z}$ action on $\mathbb{C} - 0$ with quotient the complex torus T^2, this $\mathbb{Z}$ action clearly commutes with $\mathbb{C}^\times$ action, so we get an induced holomorphic T^2 bundle $\overline{L}$ over M. The total space of $\overline{L}$ is just $C \times S^1$ and the simple degree 1 Reeb tori are the fibers of $\overline{L}$ over M. The Reeb manifolds R_a in this case are copies of M consisting of S^1-equivalence classes

of action a Reeb orbits. If $[o]$ is non-torsion, with $\langle \lambda, [o] \rangle = a$, then the moduli space $\overline{\mathcal{M}}_1(J^\lambda, A_o)$, is clearly identified with $\widetilde{R}_a$; its elements degree $\frac{a}{2\pi}$ covering maps of the simple degree 1 Reeb tori corresponding to elements $o \in R_{2\pi}$. By the discussion in the above paragraph, for each $[u] \in \widetilde{R}_a$, corresponding to $[o] \in R_a$ the corresponding Cauchy-Riemann operator D on N_u is (naturally identified with) the Dolbeault operator for the trivial holomorphic vector bundle. More specifically we have by discussion above a holomorphic identification of a normal neighborhood of the simple Reeb torus with $(U \subset T^{\mathbb{C}}_{m_o} M) \times T^2$, for some $U \ni 0$, where $m_o \in M$ denotes the point whose fiber in $\overline{L}$ is the simple degree 1 Reeb torus corresponding to o. And so $\ker D$ on N_u is identified with $T^{\mathbb{C}}_{m_o} M$. Clearly the orbifold $\widetilde{R}_a$, $\frac{a}{2\pi} \geq 1$ is the (non-effective) orbifold quotient of the smooth manifold M by the trivial action of $\mathbb{Z}_{\mathrm{mult}(o)}$. In particular $\chi(\widetilde{R}_a) = \frac{1}{\mathrm{mult}(o)} \chi(M)$.

We may then proceed as in the Proof of 1.12. Specifically, we identify the D-cokernel bundle over $\widetilde{R}_a$ with the cotangent bundle of $\widetilde{R}_a$. And so we conclude that

$$\mathcal{E}(C \times S^1, A_o) = -\chi(\widetilde{R}_a) = -\frac{1}{\mathrm{mult}(o)} \chi(M),$$

from which the first part of the theorem follows.

To prove the second part of the theorem, we note that the moduli space $\overline{\mathcal{M}}_1(J^\lambda, A_o)$ is then identified with $\sqcup_{k \geq 0} \widetilde{R}_{(1+kn)2\pi}$, where n is the order of the image of $inc_* : \pi_1(S^1) \to \pi_1(C)$. We still have that $\widetilde{R}_{(1+kn)2\pi}$ is the orbifold quotient of the smooth manifold M by the trivial action of

$$\mathbb{Z}_{1+kn}.$$

By the proof of the first part of the theorem, the obstruction bundle to each component $\widetilde{R}_{(1+kn)2\pi}$, is the (orbifold) cotangent bundle $T^*M/\mathbb{Z}_{1+kn}$. It clearly follows from this that if M admits a vector field with isolated non-degenerate zeros with positive index then on each component $\widetilde{R}_{(1+kn)2\pi}$ we may construct an abstract perturbation so that $\widetilde{R}^{vircycle}_{(1+kn)2\pi}$ is a 0-dimensional oriented weighted branched manifold with negatively signed elements, and so λ is of negative infinite type. □

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